Lattice studies of QCD-like theories with many fermionic degrees of freedom

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Abstract

I give an elementary introduction to the study of gauge theories coupled to fermions with many degrees of freedom. Besides their intrinsic interest, these theories are candidates for nonperturbative extensions of the Higgs sector of the Standard Model. While related to QCD, these systems can exhibit very different behavior from it: they can possess a running gauge coupling with an infrared attractive fixed point (IRFP). I briefly survey recent lattice work in this area.

I. INTRODUCTION

In the Standard Model of elementary particle physics, the electroweak coupling constants are small and much of the phenomenology is perturbative. The strong interactions are non-perturbative: quarks are permanently confined. But what about the unseen sector of the Standard Model, which is responsible for electroweak symmetry breaking. Is it weakly interacting, described by one or more fundamental Higgs fields, or is it also strongly interacting? The textbook Standard Model assumes it is the former. However, the idea that there is new strongly interacting dynamics in the Higgs sector is long lived. The main motivation for this is the hierarchy or "naturalness" or "fine-tuning" problem: the mass of the Higgs (or of any fundamental scalar field) depends quadratically on the ultraviolet cutoff Λ, mass at the cutoff scale m_0 and λ the bare Higgs self-coupling,

$$m_H^2 = m_0^2 + \frac{3}{4\pi} \lambda \Lambda^2 \tag{1}$$

To achieve a Higgs mass which is much (much) less than Λ requires a delicate cancellation among the various bare quantities. If the Higgs is composite, the fine tuning problem is softened. For example, if the Higgs is a bound state of some new fermions, the quadratic dependence on Λ is transformed (at one loop) into $M_H \sim \Lambda \exp(-c/g^2(\Lambda))$, or a value near the scale where the dynamics which forms the Higgs becomes strong.

The dynamics which gives the W and Z their masses is that they "eat" Goldstone bosons which were present in the ungauged theory. Any kind of Goldstone can be eaten, and the pionic analog of the new fermionic bound states makes a tasty meal. In the Lagrangian, the vacuum expectation value v of the fundamental scalar field is replaced by the pseudoscalar decay constant of the bound state, F_π, which must be set (by hand) to its real world value of $v \sim F_\pi \sim 250$ GeV. This particular idea is called "technicolor" [1, 2]. (For a review, see [3]. A recent introduction to the (continuum) theoretical background is Ref. [4].)

Lattice techniques have been very successful at dealing with the nonperturbative dynamics of QCD and indeed the lattice is now the source of many high precision calculations of strong interaction masses and matrix elements. It is natural to think that lattice techniques can be applied to strongly interacting dynamics which may appear at the energy of the Large Hadron Collider or beyond.

Before going to the lattice, let us think analytically. Suppose we have an $SU(N_c)$ gauge theory with N_f flavors of fermions of mass m, in representation R. The Gaussian fixed point at $g^2 = 0$, $m = 0$ is well understood perturbatively. The mass is a relevant operator and it will presumably remain so even at strong gauge coupling. $m = 0$ is a critical surface and we want to investigate the running of the gauge coupling along it. The analysis can begin in perturbation theory[5]. The gauge coupling's beta function is

$$\beta(g^2) = \frac{dg^2}{d\log q^2} = -\frac{b_1}{16\pi^2} g^4 - \frac{b_2}{(16\pi^2)^2} g^6 + \cdots , \tag{2}$$

where

$$b_1 = \frac{11}{3} C_2(G) - \frac{4}{3} N_f T(R) \tag{3}$$

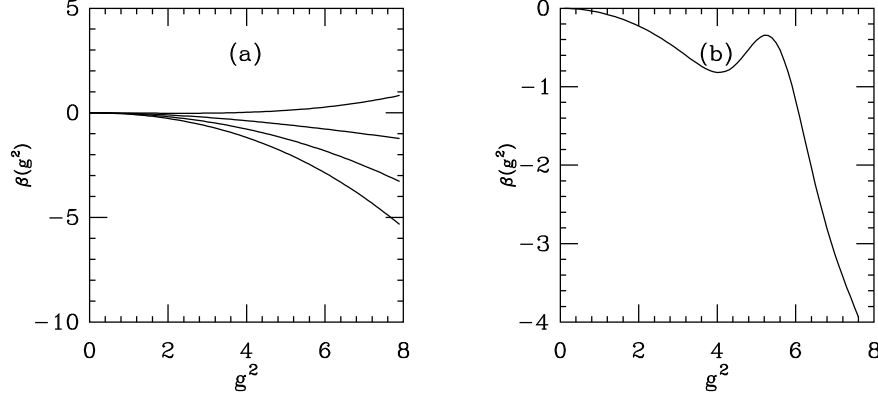$$b_2 = \frac{34}{3} [C_2(G)]^2 - N_f T(R) \left[\frac{20}{3} C_2(G) + 4C_2(R) \right] . \tag{4}$$

2

FIG. 1: Examples of beta functions. Panel (a) shows the two-loop beta function for $SU(3)$ gauge theory with (from the bottom) $N_f = 2$, 6, 10, 14 flavors of fundamental representation fermions. The $N_f = 14$ beta function starts off negative before turning positive. If $N_f > 16.5$ the beta function is everywhere positive. Panel (b) shows the "walking technicolor dream" beta function, under which the coupling constant would run out quickly to about $g^2 = 5.5$, then runs slowly before taking off again.

Here $C_2(R)$ is the value of the quadratic Casimir operator in representation R (G denotes the adjoint representation, so $C_2(G) = N_c$), while $T(R)$ is the conventional trace normalization.

These theories have an ultraviolet-attractive or infrared-repulsive or Gaussian fixed point at $g^2 = 0$. In perturbation theory, three things can happen as we flow to the infrared:

- $b_1 < 0$: the gauge coupling runs to the Gaussian fixed point in the infrared. Theories that do this are "trivial."

- $b_1 > 0$, $b_2 > 0$: the gauge coupling runs to a large value in the infrared, and long distance dynamics is nonperturbative. This is the situation for ordinary QCD ($N_c = 3$ with $N_f = 2$ or 3). In that case, the theory is confining and chiral symmetry is broken spontaneously.

- $b_1 > 0$, $b_2 < 0$: In this case it is possible that there is a critical coupling g_*^2 where $\beta(g_*^2) = 0$. The coupling runs to this value and sticks there. One says that there is an "infrared attractive fixed point" or IRFP. The range of values of N_c and N_f for which this occurs is called the "conformal window." Some examples of beta functions are shown in Fig. 1.

Conventional strong-coupling models for the Higgs system require theories which make the confinement – chirally broken choice. They have a set of Goldstone bosons which are fermion bound states, and which are food to be eaten by the electroweak gauge bosons. The Higgs particle is the analog of the sigma meson, a mesonic bound state with scalar quantum numbers. There are also towers of bound states which either have to be explained away if they are light enough to have already been discovered, or are potential "new physics" for the Large Hadron Collider.

Theories with an IRFP are like condensed matter systems possessing second order phase transitions, at their critical points. They do not have a particle spectrum: all correlation functions decay algebraically at long distances. (Georgi [6] coined the term "unparticles"

to describe the excitations to particle physicists.) For the most part, these theories are not taken to be candidates for beyond - Standard Model physics but they do occupy a leading role in theoretical particle physics: they are the CFT (conformal field theories) of AdS-CFT which are widely used as avatars for analog calculations of strong-interaction dynamics.

Not just any confining theory is a candidate for beyond - Standard Model physics. In addition to generating gauge boson masses, the Higgs sector is responsible for the masses of the ordinary fermions. Technicolor was invented to give the gauge bosons their masses. To generate fermion masses, one needs some new dynamics ("extended technicolor") at some even higher scale. It is a delicate business to simultaneously generate acceptable fermion masses while avoiding too-large flavor changing neutral currents. Successful phenomenology appears to require "walking:" which (to quote T. Appelquist [7]) is "a theory that is outside the conformal window, but close to it, with the would-be IRFP somewhat super-critical," so that the running coupling evolves very slowly over a wide range of scales.

In addition, phenomenology makes a special demand on the running of the mass parameter. The observable of interest is the anomalous dimension γ_m of the mass operator $\bar{\psi}\psi$.

$$\mu\frac{\partial m(\mu)}{\partial \mu} = -\gamma_m(g^2)m(\mu).$$ (5)

For future reference, in lowest order in perturbation theory,

$$\gamma_m = \frac{6C_2(R)}{16\pi^2}g^2.$$ (6)

Successful technicolor models typically need $\gamma_m \sim 1$ while the coupling walks [8]. Scenarios for how this may be achieved typically depend on a running gauge coupling like the one shown in Fig. 1b. Starting at some extremely high scale where the coupling is small, it runs quickly out to some intermediate low scale, where the curvature of the beta function up to near zero causes the coupling to stall. At the same time γ_m is supposed to run out to a large, nonperturbative value.

New physics at a high energy scale can influence low energy observations by inducing extra operators in the low energy effective theory. These new operators can, in turn, affect precision electroweak measurements [9]. It could happen, that one's proposed new dynamics might already be ruled out by them.

If the system is conformal at zero fermion mass m, then near $m = 0$ the correlation length ξ scales as

$$\xi \sim m^{-1/y_m}$$ (7)

where $y_m = 1 + \gamma_m(g_*^2)$ is the leading relevant exponent of the system (in the language of critical phenomena).

Unitarity bounds for conformal field theories[10, 11] constrain the scaling dimension of the leading scalar operator to lie in the range $3 > d = 4 - y_m > 1$. For our technicolor candidates, this operator is $\langle \bar{\psi}\psi \rangle$, the "techni-" condensate, nonzero because a nonzero fermion mass explicitly breaks chiral symmetry. At the top of the conformal window, $y_m = 1$ its free field value. For large N_c and N_f it is possible (see Ref. [11]) to tune the N_f/N_c ratio so that g_* is order ϵ and then $\gamma_m \sim \epsilon$ and $y_m = 1 + O(\epsilon)$. It is believed that d decreases, or γ_m increases, as one moves to the bottom of the conformal window.

Nothing is known about how the conformal window closes, or more mundanely, what happens at the conformal – chirally broken and confining boundary. The only way I understand this question is by imagining some continuously variable parameter which can carry

one across the boundary. The value of N_f/N_c at large N_c may be such a parameter. One possibility (which is what happens in Fig. 1(a)) is that a zero of the beta function "walks in" from infinite g^2. Another possibility, inspired by Fig. 1(a), is that a single new fixed point just appears on the real axis, coming out of the complex g^2 plane, and then splits into an IRFP (the conformal fixed point) and a second UV fixed point at larger g^2 [12]. There is also an extensive literature relating the lower end of the conformal window to large y_m, with $y_m = 2$ perhaps having a connection with physics which closes the window. (See Refs. [13–17].) To me, this connection is Eq. 7, which, when the correlation length is replaced by $1/M$ for some (the pseudoscalar?) mass, becomes

$$M^{y_m} \propto m_q. \tag{8}$$

The value $y_m = 2$ is (coincidentally?) the Gell-Mann, Oakes, Renner relation relevant to the additional explicit breaking (by the fermion mass) of spontaneously broken chiral symmetry.

Thus, a rich program presents itself to anyone who wants to look for nonperturbative beyond - Standard Model physics:

- Where does any particular theory sit, inside or outside the conformal window?

- What is the value of γ_m, either as a function of g^2 or at any fixed point?

- If the theory is confining, what is its particle spectrum? What is the Higgs mass and what are its couplings?

- Does the theory survive precision electroweak constraints, or predict observable deviations from pure Standard Model predictions?

Already the first point is challenging. Fig. 2 shows a phase diagram from an approximate analytic calculation from Ref. [18]. This figure has served as the target for many lattice calculations.

II. BEYOND PERTURBATION THEORY

We do not have to restrict ourselves to a perturbative analysis of these theories. instead, let us think about them as we would any critical statistical system, using the language of fixed points, critical surfaces, and renormalization group flow. (For a related discussion, see Ref. [19].)

A theory like QCD, labeled with a gauge coupling constant g^2 and a single fermion mass m, has a Gaussian fixed point at $(g^2, m) = (0, 0)$ (see the left panel of Fig. 3). Under a sequence of real space blocking transformations (flows to the IR, in the language of the last section) both the running coupling and the mass will grow. To show that flow, I have drawn arrows in the figure. Of course, the most general theory which one can write down at the cutoff scale will also contain irrelevant couplings. The critical surface is the surface of $(g^2, m) = (0, 0)$ projected onto the space of all bare couplings, and under blocking to the IR, flows beginning on the critical surface will carry the coupling to some critical point on the critical surface. Off the critical surface, flows carry us ever farther away from it.

The analog of the preceding figure for the case of a theory with an IRFP is shown in Fig. 4. Again the mass is relevant. However, this time the gauge coupling flows to an IRFP, meaning that the gauge coupling itself is irrelevant (more precisely, $g^2 - g_*^2$ is an irrelevant
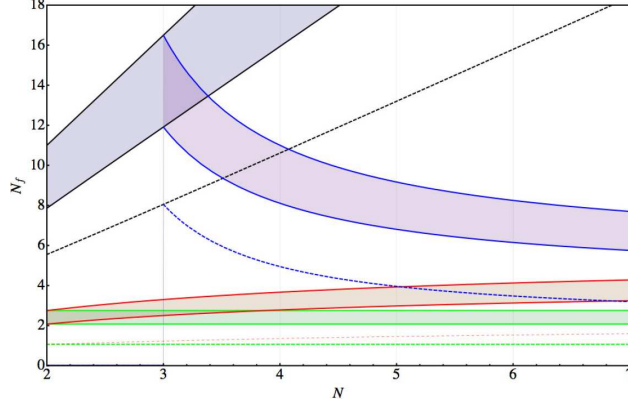
FIG. 2: Conjectured phase diagram (due to Ref. [18]) for nonsupersymmetric theories with fermions in the: i) fundamental representation (blue), ii) two-index antisymmetric representation (purple), iii) two-index symmetric representation (red), iv) adjoint representation (green) as a function of the number of flavors N_f and the number of colors N. The shaded areas depict the corresponding conformal windows from their calculation. The dashed curve represents the change of sign in the second coefficient of the beta function.
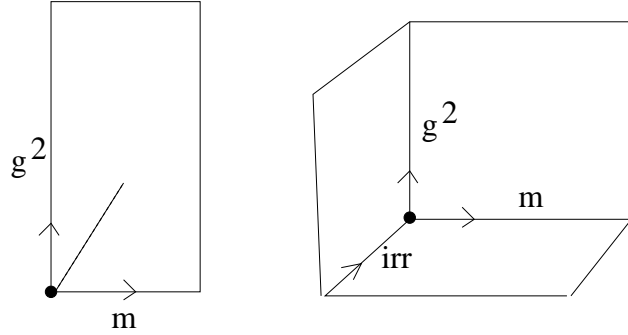

FIG. 3: Schematic coupling constant space of a QCD-like theory. The left panel shows only the relevant mass and marginally relevant gauge couplings, with IR flow lines and a renormalized trajectory. The right panel adds a single irrelevant direction.

direction). The critical surface includes part of the g^2 axis. Precisely at $m = 0$ the theory is critical; all correlation functions decay algebraically.

Since the mass is relevant, RG flow carries us off the $m = 0$ axis. Since it is the relevant perturbation, it controls the correlation length through Eq. 7. It is easiest to express the correction to the free energy (D is the dimension, u_i is any irrelevant operator),

$$
\begin{aligned}
f_s(m, u_i) &= m^{D/y_m} f_s(m_0, u_{i0} + (u_i - u_{i0}) \left(\frac{m}{m_0}\right)^{|y_i|/y_m}) \\
&= m^{D/y_m}(A_1 + A_2 m^{|y_i|/y_m}),
\end{aligned}
$$

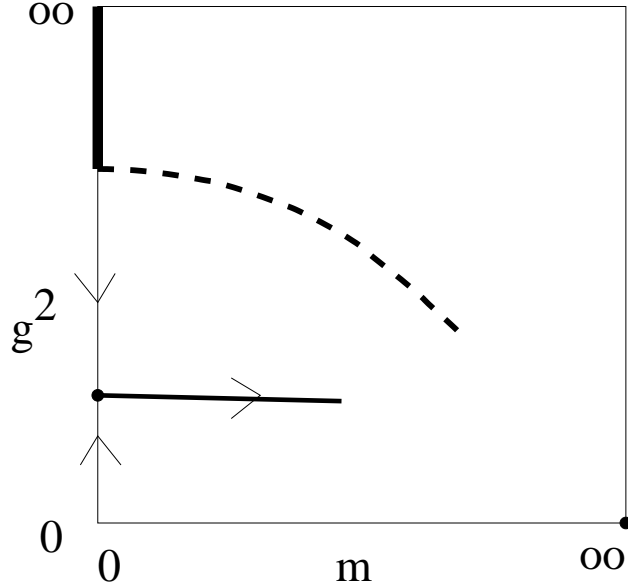(9)

where A_1 and A_2 are non-universal constants.

FIG. 4: A schematic picture of the parameter space of a theory with an IRFP. The part of the $m = 0$ axis drawn with a thin line is on the critical surface containing the IRFP. I have also put a dot at $(g^2, m) = (0, \infty)$ for the pure gauge theory. The thick lines show possible strong coupling crossover or transition lines. The line emanating from the IRFP is a schematic renormalized trajectory.

Fig. 4 has additional features. The pure gauge theory is confining, and so there is a second UV fixed point, located at $(g^2, m) = (0, \infty)$. The presence of a nonzero mass means that the theory has a mass gap, and so it is easy to conjecture that when the mass becomes large, the system "heals over" to a pure gauge theory, which also has a mass gap. Whether there is an additional transition associated with this behavior is unknown. In addition, many lattice regulated gauge plus fermion theories are known to confine at strong coupling. If there is end of the critical surface, drawn as the solid line in Fig. 4, there must be an additional transition, but as far as is known, nothing forces it to have any particular order.

Finally, we can add a third direction for all the irrelevant operators present in a lattice calculation. This produces the horrible Fig. 5. I show it to make the point that lattice simulations are done with a lattice action which of necessity, contains irrelevant operators. We flow onto the critical surface with enough RG evolution, but getting enough aspect ratio to complete the flow in an actual lattice simulation may not be possible.

Thus, we have a second way of viewing the difference between a confining and a conformal theory, and the suggestion that if the latter case exists, the phase diagram might be more complicated than for a confining theory. It also introduces a new concept: for an IRFP theory, the gauge coupling itself is an irrelevant coupling, and the way it affects physics, basically Eq. 9, is quite different from the way the gauge coupling in a confining theory does.

Finally, the location of an IR repulsive fixed point on a relevant axis is physical: one must tune the relevant coupling carefully to approach criticality. In all other directions its location is scheme dependent: different choices for blocking transformations move it around. By design, the IRFP has been tuned to lie someplace on a critical surface (we have set $m_q = 0$), so "where it is" is completely scheme dependent. In simulations, if the relevant
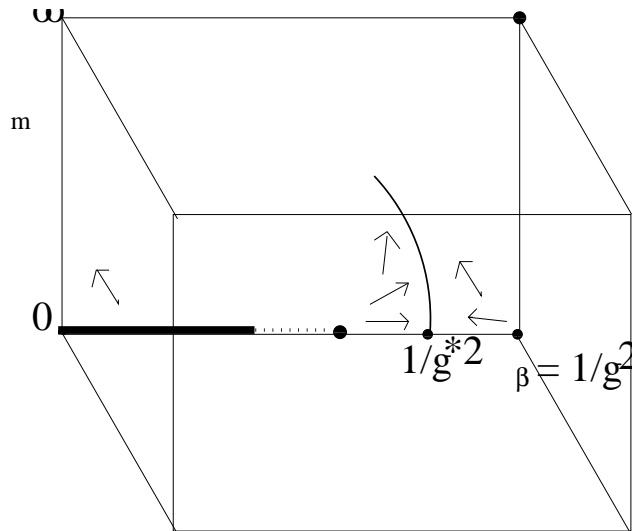
FIG. 5: Extending Fig.4 by adding an irrelevant direction.

quark mass is nonzero, the correlation length is finite. But the variation of the correlation length (which is a spectral observable) on the gauge coupling is no different from the behavior of any physical observable on an irrelevant operator. What is physical are the exponents, the derivatives of the beta functions at the critical point. For a diagonal set of couplings these are (μ is a UV scale change)

$$\mu\frac{\partial u_i}{\partial \mu} = -y_i(u_i - u_i^*). \tag{10}$$

Simple dimensional counting suggests $y_m \sim 1$ for the mass and $y_g \sim 0$ for the gauge coupling.

III. PRACTICAL PROBLEMS FOR LATTICE SIMULATIONS

At this point the lattice method for studying a beyond - Standard Model theory seems straightforward: take a QCD code, change the "3" of three colors to some other number, write subroutines for either many flavors of fermions or for fermions in higher representations, pick a simulation volume, collect data . . . . Life is, however, not so simple, and all simulations done to date are plagued by two kinds of systematic effects. In the field they are called "cutoff effects" and "finite size effects."

"Cutoff effects " describe the problem that the specific lattice action which is simulated is a bare action defined at the cutoff scale, at a distance of one lattice spacing. Calculations in quantum field theory are only real predictions when the cutoff has been removed from the calculation. In lattice QCD simulations, this is referred to as "taking the continuum limit," and it is done by tuning the bare g^2 to the Gaussian fixed point. The lattice spacing is then traded in for a Λ parameter, and dimensionful predictions can be made in units of Λ. Typically, in any lattice version of an asymptotically free theory, cutoff dependence is strongest when the bare gauge coupling is strong.

"Finite size effects" arise when the finite simulation volumes used for numerical studies force the system to have different behavior than it would in an infinite system. The classic

example of a finite size effect which is physical is when the temporal length L_t of the lattice is finite. Its length corresponds to a finite temperature $T \propto 1/L_t$. For a confining theory, a simulation volume which is smaller than the physical confinement scale will alter the spectrum and also alter the phase of the vacuum. The system size rules the dynamics.

In a confining theory with a UV attractive fixed point, the lattice spacing shrinks as the bare gauge coupling is reduced, and so if the size of the simulation volume is not increased, one is studying the theory in a box of small physical size. The squeezed system may have completely different properties from what it would have in a bigger volume. This is the situation for QCD.

These difficulties merge in the systems which have been studied to date. The problem is that, having an IRFP or not, their running coupling runs very slowly. This occurs because the leading coefficient of the beta function, b_1 in Eq. 4, is small. At one loop, the running coupling is

$$\frac{1}{g^2(sL)} = \frac{2b_1}{16\pi^2} \log sL + \text{constant.} \tag{11}$$

For ordinary QCD with two flavors, $b_1 = 29/3$. Your author has worked on $N_c = 3$ color with $N_f = 2$ symmetric representation fermions, where $b_1 = 13/3$. This slow running means that for any practical available simulation volume, the only way to have a theory which is strongly interacting at long distance is to make it strongly interacting at short distance, and thus to risk being contaminated by cutoff effects.

The slow running also means that a small change in the bare coupling can correspond to a large change in the physical scale. This can be seen by differentiating Eq. 11:

$$\Delta \frac{1}{g^2(sL)} = \frac{2b_1}{16\pi^2} \frac{\Delta s}{s} \tag{12}$$

For an asymptotically free but slowly walking theory, a tiny change in the bare coupling corresponds to a greater and greater change of scale.

Of course, if the system has an IRFP, this paragraph makes no sense. The gauge coupling is irrelevant. Only the amount by which the bare mass is tuned away from zero can affect the correlation length – asymptotically. As long as the correlation length is finite, tuning g^2 will alter it, just as tuning any irrelevant operator (like an irrelevant coefficient in a lattice action) will change the correlation length. The problem, of course, is distinguishing this "irrelevant change of length with coupling" from the true change of length which occurs in a confining and asymptotically free theory. So we are back to our fundamental problem: how to tell "no running" (IRFP) from "slow running" (Gaussian fixed point and nothing else) in a noisy lattice Monte Carlo environment with both a nonzero UV cutoff and a finite simulation volume.

This is very different from ordinary QCD. With its rapidly running coupling, one can be weakly interacting at short distance and strongly interacting at long distance, in the same simulation volume. This can be observed in (for example) a heavy quark potential behaving as $V(r) \sim 1/r$ at small r and σr for large r.

However, a bug can be a feature. Slow running is almost no running. It is possible to use the finite volume as a diagnostic. One way to do this is to adapt the familiar finite-size scaling analysis of critical phenomena. Here the length scale associated with the simulation volume L (actually $1/L$) is a relevant operator and if the correlation length scales as $\xi \sim (1/m)^{-1/y_m}$ in infinite volume, then it scales in finite volume proportionally to L times some arbitrary

function of the dimensionless ratio ξ/L

$$\xi_L = LF(\xi/L) = Lf(L^{y_m}m_q).\tag{13}$$

Results from many L's can be combined to extract y_m. To the particle physicist, the inverse correlation length could be a mass, or any other dimensionful derived quantity (a decay constant, for example).

Another way involves using L itself as a scale for a coupling $g^2(L)$ which is derived from the expectation value of some observable. Fixed L can be combined with simulations with fixed boundary conditions to give the "Schrödinger functional" suite of techniques, which can be used to measure a running coupling and γ_m [20–26]. I pause to describe this in more detail:

The basic idea is to define a running coupling $g^2(L)$ through the response of a system in a box of size L to its environment. In a gauge theory this is done by doing simulations in a finite box of size L and fixing the value of the spatial link variables on the faces of the box at Euclidean time $t = 0$ and $t = L$. The boundary conditions involves a free parameter η. Call the resulting partition function the Schrödinger functional

$$Z(\eta) = \int [dU]\exp(-S(U,\eta)).\tag{14}$$

The coupling is then defined through the variation of the effective action Γ, the negative logarithm of the partition function $Z(\eta)$: $\Gamma = -\ln Z(\eta)$. In lowest order perturbation theory, Γ is equal to the classical action, which can be computed since the link variables simply interpolate between their boundary values. At this order the action is proportional to the inverse squared bare coupling $1/g_0^2$. The renormalized coupling $g^2(L)$ is defined through

$$\frac{\partial\Gamma}{\partial\eta} = \frac{k}{g^2(L)},\tag{15}$$

where the constant k is adjusted so that $g^2(L) = g_0^2$ in lowest order. For a pure gauge system the quantity $\frac{\partial\Gamma}{\partial\eta}$ is an expectation value of gauge field variables on the boundaries.

Next we perform a second simulation in a volume of size sL_0 and compute the change in the coupling

$$\int_{L_0}^{sL_0}\frac{dL}{L} = \int_{g^2(L_0)}^{g^2(sL_0)}\frac{dg^2}{\beta(g^2)} \equiv \int_u^{\sigma(s,u)}\frac{dv}{\beta(v)},\tag{16}$$

where the "step scaling function", $\sigma[s, u = g^2(L)] = g^2(sL)$, is the new coupling constant. The running coupling is found by doing simulations with the same bare coupling on systems of size L_0 and sL_0, and, by measuring u and $\sigma(s,u)$, to see how the new coupling depends on the original one. The "discrete beta function" is $\sigma(s,u) - u$.

For asymptotically free theories, one can repeat the matching with various values of L, tuning the bare parameters to stay at fixed u as L is changed. Lattice artifacts can be removed by comparing the discrete beta function for different values of L. When the coupling runs quickly enough, one can "daisy chain" several L_0 to sL_0 pairs to see running over a scale factor s^n. This can be done directly in ordinary QCD, but with beyond Standard Model theories, the coupling runs too slowly to do this. Instead, simulations either stop with the discrete beta function or plot parameterizations of the data.

In another approach [27–29], one can attempt to walk onto the renormalized trajectory by performing simulations on lattices of various size, blocking the variables, and matching

simulations after different levels of blocking. Here the schemes basically corresponds to the choice of blocking transformation. For example, if a simulation on one lattice size L_1 at some g_1 and blocked n_b times with a blocking scale b produces the same observable as a simulation on lattice size $L_2 = L_1/b$ does after $n_b - 1$ blocking steps, we would say that the bare couplings must change from g_1 to g_2 when the cutoff changes from a to ba. Running of bare quantities holding physical quantities fixed defines a (finite scale) beta function. This is called the "Monte Carlo Renormalization Group."

Essentially all published results for running g^2 and γ_m come from one of these methods. Note that they do not depend on the numerical simulations being done at large physical volume or that the equilibrium state is in some particular phase.

And there are some explicitly lattice - related problems.

The first comes from the choice of lattice fermions [30]. To make a long story short, there are three kinds of lattice fermions. The first ones are Wilson or clover fermions. They explicitly break chiral symmetry due to the presence of lattice operators which prevent "fermion doubling." However, interactions give an additive renormalization to the bare fermion mass and finding $m_q = 0$ (where the quark mass is defined through some analog of continuum chiral symmetry, like the PCAC relation, $\partial_\mu A^\mu(x) = 2m_q P(x)$ relating the axial current A_μ, the pseudoscalar current P and the quark mass m_q) involves fine tuning bare lattice quantities.

The second choice is "staggered fermions." These naturally come in multiples of four flavors (called "tastes" in the literature) and, to make a long story short, have an exact $U(1)$ chiral symmetry while the expected $SU(4) \otimes SU(4)$ chiral symmetry is broken by lattice artifacts. This is called the "flavor problem" or "doubling problem." The conversion of four flavors into an arbitrary number is well understood in QCD [31], but (as far as I know) only if the theory is chirally broken are there observables which monitor flavor symmetry breaking, namely the masses of the would-be Goldstone bosons. They are nondegenerate at any lattice spacing and nonzero (apart from one pion) at $m_q = 0$.

Finally, there are chiral fermions, overlap or domain wall fermions. They are quite expensive and either fail to be completely chiral or become too expensive to simulate when the gauge field configurations become rough (at strong coupling).

With a small number of flavors, the strong coupling limit of lattice theories with all three kinds of fermions is confining and chirally broken. There are pseudoscalars whose squared mass vanishes with the quark mass, and it is possible to tune bare parameters so that they vanish. However, with a large number of fermionic degrees of freedom, for example $N_f \sim 7$ flavors in color $SU(3)$, Wilson fermions develop a first order transition at strong coupling and $m_q = 0$ cannot be reached in a stable ground state: the quark mass jumps discontinuously from positive to negative [32]. In finite volume (where all simulations are done) this transition becomes entangled with the finite temperature confinement - deconfinement transition.

The most unambiguous situation, from the point of view of a simulation, would be to observe confinement and chiral symmetry breaking at strong and intermediate coupling, a beta function which is everywhere negative (so the coupling grows under flow to the IR) at weak and intermediate coupling, and a region of overlap where both a negative beta function and confinement are both observed. In this case we have a situation like low-N_f QCD, where the continuum theory is asymptotically free, confining and chirally broken.

IV.   THE STATE OF THE ART – EARLY AUTUMN 2010

The cost of $N \times N$ matrix multiplication scales like N^3 and so most lattice simulations use $N_c = 2$ or 3. Referring to Fig. 2, there are then two ways to approach the conformal window: either by increasing N_f while fixing N_c and holding the representation fixed (to the fundamental, so far), or by fixing $N_f = 2$, increasing N_c and going to ever higher-dimensional representations. I will call these two choices the "vertical" and "horizontal" approaches.

There are many simulations of $N_c = 2$ and 3 along the vertical axis. For $N_c = 3$ and $N_f \leq 8$ the situation seems uncontroversial: The beta function is negative throughout the range of couplings where it is measured [33, 34], and confinement and chiral symmetry breaking are observed in conventional simulations [35]. These theories could be technicolor candidates, except that their couplings apparently run too fast to satisfy phenomenological constraints (they do not "walk" and they fail precision electroweak tests). A recent set of large scale simulations [36] has tried to observe the onset of "condensate enhancement." The idea is that, associated with a large γ_m, the condensate, expressed dimensionally as $\langle \bar{\psi}\psi \rangle / F_\pi^3$, should grow compared to the (low-N_f) QCD case. Precision electroweak comparisons have also begun [37].

Large N_f presents a difficult problem in the chiral limit, because one-loop corrections to the lowest order formulas for the condensate and F_π are proportional to N_f. These one-loop corrections are also the source of finite volume corrections to chiral observables. Thus large N_f simulations become quite sensitive to the simulation volume. This complicates extrapolations to the zero fermion mass limit.

Above $N_f = 8$ the situation is ambiguous. The original studies of Appelquist, Fleming and Neil [33, 34] saw an IRFP for $N_c = 3$, $N_f = 12$. The problem is that the authors of Refs. [38, 39] observe that systems with 8 to 12 flavors of (staggered) fermions appear to be chirally broken and confining. This is incompatible with the presence of an IRFP unless there is an additional transition which marks the boundary of the confining phase with the critical surface of the IRFP. Perhaps [40] this transition has been observed. Again the difficulty is in determining if the transition is induced by the finite simulation volume or if it persists in infinite physical volume.

Finally, an $SU(2)$ simulation with 6 flavors of Wilson fermions[41] claims an IRFP very close to the first order line, with a large $\gamma_m \sim 0.7$ with a large uncertainty (presumably because the transition is near the first order line).

Now for the horizontal branch. All simulations not using fundamental representation fermions use symmetric-representation ones. Looking at Fig. 2, we expect that $N_f = 2$ systems are close to conformal, with perhaps larger N_c more likely to be confining.

Many groups (a representative list includes [42–47]) have studied $N_f = 2$ flavors of adjoint representation fermions in $SU(2)$. All studies to date use Wilson type quarks. The Schrödinger functional coupling runs very slowly throughout the weak coupling phase. Two groups [44, 45] claim evidence for an IRFP at strong bare gauge coupling. With Wilson quarks, the $m = 0$ line in bare coupling space collides with a line of first order transitions at strong coupling. The collision point appears to be an IR-repulsive critical point, giving the $m = 0$ system two UV repulsive fixed points (at $g = 0$ and at large g). The IRFP is found very close to the end of the first order line. It is unknown whether this compromises the results.

$SU(3)$ with $N_f = 2$ symmetric representation fermions is similar. An earlier claim of an IRFP by Shamir, Svetitsky and DeGrand [48] was not confirmed by a subsequent simulation

by the same authors with a better lattice action[49]. The running coupling just runs very slowly.

Fortunately, the slowly running gauge coupling makes the nearly conformal theory "conformal for all practical purposes:" that is, at any value of the bare coupling, the coupling runs so little that data can be analyzed as if it did not run at all. Then the mass is the relevant perturbation and all the statistical mechanics machinery for scaling near a critical point can be employed. This gives $\gamma_m(g^2)$ for $SU(2)$ in a wide variety of ways[45–47]. It turns out to be small, less than about 0.5, and consistent with perturbation theory, Eq. 6. Two measurements in $SU(3)$ with sextet fermions [49, 50] of γ_m exploiting this fact gave $\gamma_m < 0.6$ over the observed range, again perturbative. Here we also have a calculation of a precision electroweak observable [51], which does not look like a phenomenologist's dream for viable technicolor.

The two cases of difficult running coupling and easy γ_m are illustrated by plots from the author's collaboration's study of $SU(3)$ gauge theory with $N_f = 2$ flavors of sextet fermions. See Figs. 6 - 7.

V.   CONCLUSIONS

It's a fascinating business, trying to use lattice simulations to search for and analyze nearly conformal quantum field theories. People do not agree on what are the important questions, or how to answer them.

The main problem with simulations in this field is that the coupling constant runs so slowly that by the time one forces the theory to be strongly interacting at long distance, it is strongly interacting at short distance and so not easily related to a continuum action. Perhaps there is some way to overcome this problem, by clever "lattice action design." Failing that, it may still be possible to settle the question of whether these near-conformal theories possess phenomenological viability. Experiment fortunately puts stringent constraints on strongly interacting beyond - Standard Model physics. For example, walking technicolor needs a large γ_m; if no model has that, then this approach to a nonperturbative Higgs sector is ruled out.

The last thing I can say: To date, no observed beta function looks like the "walking technicolor dream," Fig. 1b. They all look like simple deformations of two loop perturbation theory, like Fig. 1a.

We are still a long way from doing high precision calculations at the level of those for lattice QCD. Of course, in QCD we knew what the answer was, before we started: confinement and chiral symmetry breaking. Here we don't!
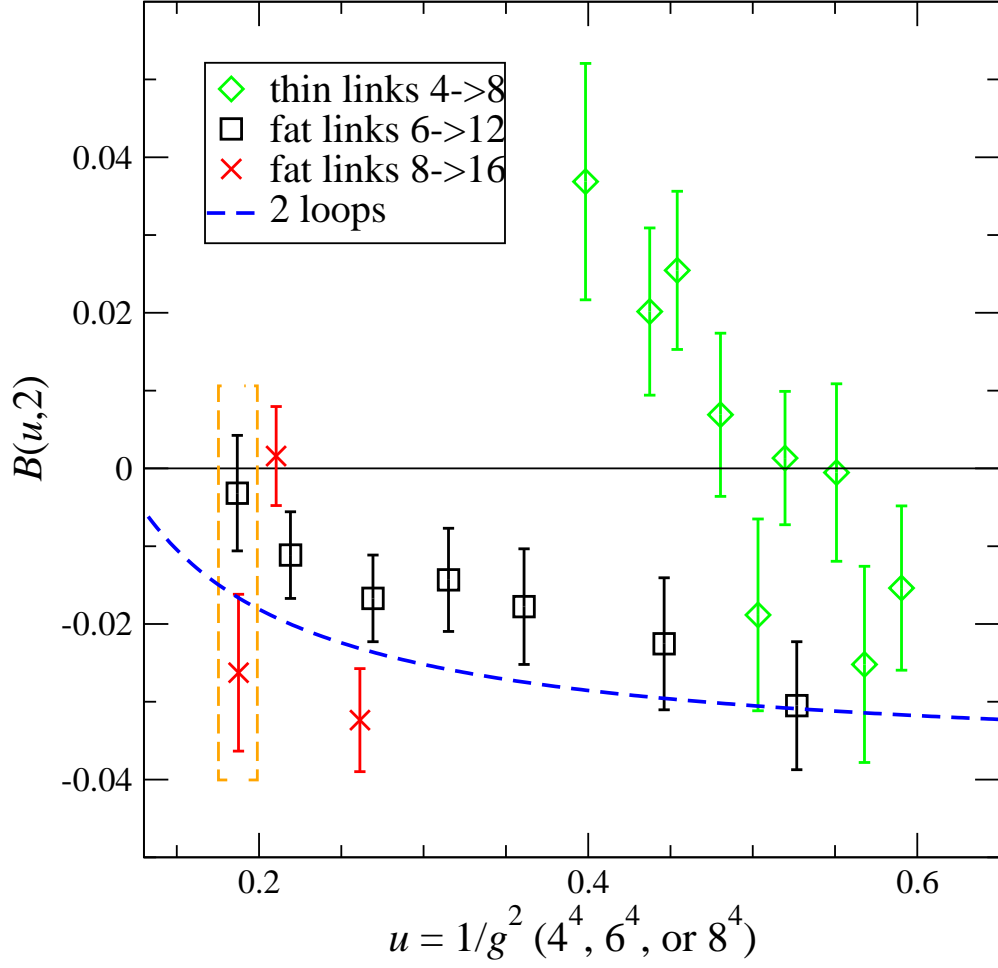
Acknowledgments

FIG. 6: Discrete beta function for the scale transformation $L \to 2L$, for the SU(3) gauge theory with sextet fermions. "Thin links" indicates the otherwise unimproved Wilson–clover action used in Ref. [48]; "fat links" indicates the use of the improved lattice action of Ref. [49]. The horizontal axis is the SF running coupling computed on the smaller lattice, while the vertical axis gives the change in $1/g^2$ in doubling the lattice size. The dashed curve is the two-loop result. The DBF would change sign at an infrared-attractive fixed point. The thin-link data show an IRFP at $g^2 \simeq 2$, while the fat-link data exclude this and show no clear evidence of a fixed point before lattice artifacts set in at $g^2 \simeq 5$.

[1] S. Weinberg, Phys. Rev. D **13**, 974 (1976).

[2] L. Susskind, Phys. Rev. D **20**, 2619 (1979).

[3] For reviews of the phenomenology, see C. T. Hill and E. H. Simmons, Phys. Rept. **381**, 235 (2003) [Erratum-ibid. **390**, 553 (2004)] [arXiv:hep-ph/0203079]; R. S. Chivukula, Lectures
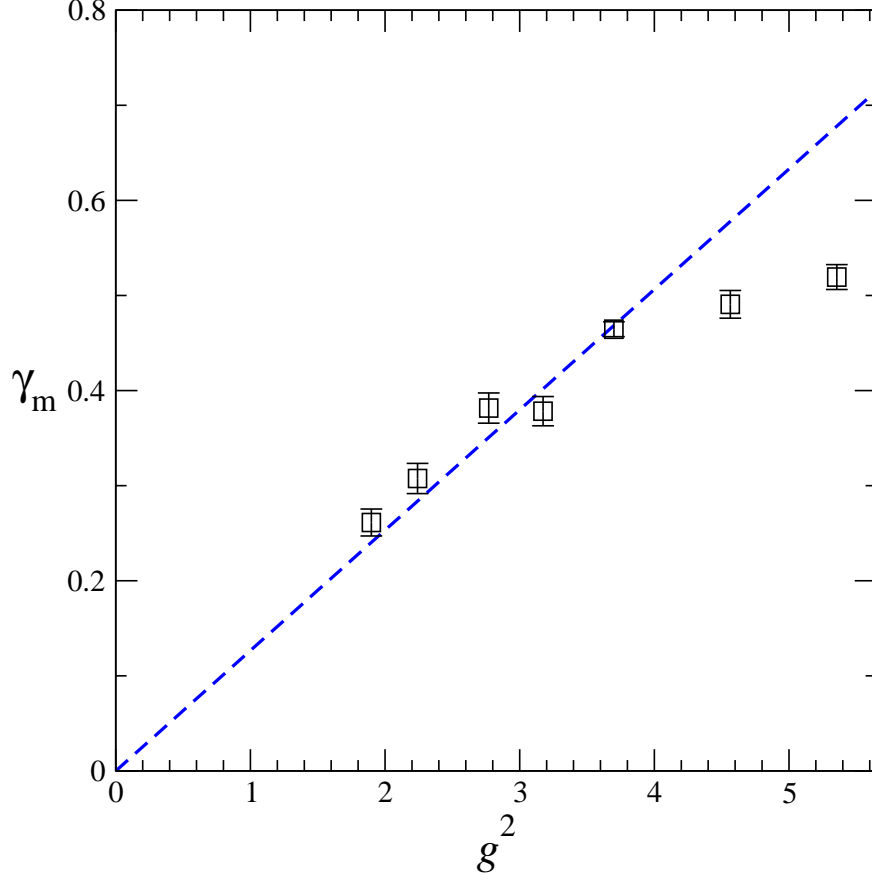
FIG. 7: Anomalous dimension γ_m in the SU(3) gauge theory with sextet fermions, determined from Ref. [49]. The dashed line is the one-loop perturbative result. As the coupling gets stronger the non-perturbative result saturates at values less than 0.6.

given at Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2000): Flavor Physics for the Millenium, Boulder, Colorado, 4-30 Jun 2000. Published in "Boulder 2000, Flavor physics for the millennium" (World publishing, 2000), J. Rosner, ed.; 731-772; [arXiv:hep-ph/0011264]

[4] R. Contino, "Tasi 2009 lectures: The Higgs as a Composite Nambu-Goldstone Boson," arXiv:1005.4269 [hep-ph].

[5] The perturbative analysis goes back to W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974) and T. Banks and A. Zaks, Nucl. Phys. B **196**, 189 (1982).

[6] H. Georgi, Phys. Rev. Lett. **98**, 221601 (2007) [arXiv:hep-ph/0703260]; Phys. Lett. B **650**, 275 (2007) [arXiv:0704.2457 [hep-ph]];

[7] Quote from a letter to the author by T. Appelquist.

[8] For a recent discussion, see R. S. Chivukula and E. H. Simmons, "Condensate enhancement and D-meson mixing in technicolor theories," arXiv:1005.5727 [hep-lat].

[9] A good recent review is W. Skiba, "TASI Lectures on Effective Field Theory and Precision Electroweak arXiv:1006.2142 [hep-ph].

[10] G. Mack, Commun. Math. Phys. **53**, 155 (1977).

[11] B. Grinstein, K. A. Intriligator and I. Z. Rothstein, Phys. Lett. B **662**, 367 (2008)

[arXiv:0801.1140 [hep-ph]].

[12] D. B. Kaplan, J. W. Lee, D. T. Son and M. A. Stephanov, Phys. Rev. D **80**, 125005 (2009) [arXiv:0905.4752 [hep-th]].

[13] A. G. Cohen and H. Georgi, Nucl. Phys. B **314**, 7 (1989).

[14] J. Braun and H. Gies, JHEP **0606**, 024 (2006) [arXiv:hep-ph/0602226].

[15] T. Appelquist, J. Terning and L. C. R. Wijewardhana, Phys. Rev. Lett. **77**, 1214 (1996) [arXiv:hep-ph/9602385].

[16] T. Appelquist, A. Ratnaweera, J. Terning and L. C. R. Wijewardhana, Phys. Rev. D **58**, 105017 (1998) [arXiv:hep-ph/9806472].

[17] E. Gardi and G. Grunberg, JHEP **9903**, 024 (1999) [arXiv:hep-th/9810192].

[18] D. D. Dietrich and F. Sannino, Phys. Rev. D **75**, 085018 (2007) [arXiv:hep-ph/0611341].

[19] T. DeGrand and A. Hasenfratz, Phys. Rev. D **80**, 034506 (2009) [arXiv:0906.1976 [hep-lat]].

[20] M. Lüscher, R. Narayanan, P. Weisz and U. Wolff, Nucl. Phys. B **384**, 168 (1992) [arXiv:hep-lat/9207009].

[21] M. Lüscher, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B **413**, 481 (1994) [arXiv:hep-lat/9309005].

[22] S. Sint and R. Sommer, Nucl. Phys. B **465**, 71 (1996) [arXiv:hep-lat/9508012].

[23] M. Lüscher, S. Sint, R. Sommer and P. Weisz, Nucl. Phys. B **478**, 365 (1996) [arXiv:hep-lat/9605038]; M. Lüscher, S. Sint, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B **491**, 323 (1997) [arXiv:hep-lat/9609035].

[24] For a pedagogic review see R. Sommer, arXiv:hep-ph/9711243.

[25] K. Jansen and R. Sommer [ALPHA collaboration], Nucl. Phys. B **530**, 185 (1998) [Erratum-*ibid.* B **643**, 517 (2002)] [arXiv:hep-lat/9803017].

[26] M. Della Morte *et al.* [ALPHA Collaboration], Nucl. Phys. B **713**, 378 (2005) [arXiv:hep-lat/0411025].

[27] A. Hasenfratz, Phys. Rev. D **82**, 014506 (2010) [arXiv:1004.1004 [hep-lat]].

[28] A. Hasenfratz, arXiv:0911.0646 [hep-lat].

[29] A. Hasenfratz, Phys. Rev. D **80**, 034505 (2009) [arXiv:0907.0919 [hep-lat]].

[30] Readers unfamiliar with the technical details of lattice calculations might consult T. DeGrand and C. DeTar, *Lattice Methods for Quantum Chromodynamics*, (World Scientific, Singapore, 2006).

[31] For reviews of the situation, see S. R. Sharpe, PoS **LAT2006**, 022 (2006) [arXiv:hep-lat/0610094]. M. Golterman, PoS C **ONFINEMENT8**, 014 (2008) [arXiv:0812.3110 [hep-ph]].

[32] Y. Iwasaki, K. Kanaya, S. Kaya, S. Sakai and T. Yoshie, Phys. Rev. D **69**, 014507 (2004) [arXiv:hep-lat/0309159]. See also K. i. Nagai, G. Carrillo-Ruiz, G. Koleva and R. Lewis, Phys. Rev. D **80**, 074508 (2009) [arXiv:0908.0166 [hep-lat]].

[33] T. Appelquist, G. T. Fleming and E. T. Neil, Phys. Rev. Lett. **100**, 171607 (2008) [arXiv:0712.0609 [hep-ph]].

[34] T. Appelquist, G. T. Fleming and E. T. Neil, arXiv:0901.3766 [hep-ph].

[35] A. Deuzeman, M. P. Lombardo and E. Pallante, arXiv:0804.2905 [hep-lat]. X. Y. Jin and R. D. Mawhinney, PoS **LATTICE2008**, 059 (2008) [arXiv:0812.0413 [hep-lat]].

[36] T. Appelquist *et al.*, Phys. Rev. Lett. **104**, 071601 (2010) [arXiv:0910.2224 [hep-ph]].

[37] T. Appelquist *et al.*, arXiv:1009.5967 [hep-ph].

[38] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, Phys. Lett. B **681**, 353 (2009) [arXiv:0907.4562 [hep-lat]].

[39] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, arXiv:0911.2463 [hep-lat].

[40] A. Deuzeman, M. P. Lombardo and E. Pallante, arXiv:0904.4662 [hep-ph].

[41] F. Bursa, L. Del Debbio, L. Keegan, C. Pica and T. Pickup, arXiv:1007.3067 [hep-ph].

[42] S. Catterall, J. Giedt, F. Sannino and J. Schneible, JHEP **0811**, 009 (2008) [arXiv:0807.0792 [hep-lat]].

[43] A. J. Hietanen, J. Rantaharju, K. Rummukainen and K. Tuominen, JHEP **0905**, 025 (2009) [arXiv:0812.1467 [hep-lat]].

[44] A. J. Hietanen, K. Rummukainen and K. Tuominen, Phys. Rev. D **80**, 094504 (2009) [arXiv:0904.0864 [hep-lat]].

[45] F. Bursa, L. Del Debbio, L. Keegan, C. Pica and T. Pickup, Phys. Rev. D **81**, 014505 (2010) [arXiv:0910.4535 [hep-ph]].

[46] L. Del Debbio, B. Lucini, A. Patella, C. Pica and A. Rago, Phys. Rev. D **82**, 014510 (2010) [arXiv:1004.3206 [hep-lat]].

[47] L. Del Debbio, B. Lucini, A. Patella, C. Pica and A. Rago, Phys. Rev. D **82**, 014509 (2010) [arXiv:1004.3197 [hep-lat]].

[48] Y. Shamir, B. Svetitsky and T. DeGrand, Phys. Rev. D **78**, 031502 (2008) [arXiv:0803.1707 [hep-lat]].

[49] T. DeGrand, Y. Shamir and B. Svetitsky, arXiv:1006.0707 [hep-lat].

[50] T. DeGrand, Phys. Rev. D **80**, 114507 (2009) [arXiv:0910.3072 [hep-lat]].

[51] T. DeGrand, arXiv:1006.3777 [hep-lat].